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LETTER TO THE EDITOR

On the $x^2+\lambda x^2/(1+gx^2)$ interaction

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Abstract. We present an infinite set of exact solutions of odd parity for the one-dimensional Schrödinger equation with the interaction $x^2+\lambda x^2/(1+gx^2)$. These complement the exact solutions of even parity that have been obtained recently by Flessas.

The existence of an infinite number of exact solutions of the Schrödinger equation

$$(\mathrm{d}^2/\mathrm{d}x^2+E-x^2-\lambda x^2/(1+gx^2)]\psi(x)=0, \qquad -\infty<x<+\infty, \qquad g>0, \tag{1}$$

with eigenfunctions given by products of exponential and polynomial functions of x^2 for specific relations between the couplings g and λ, has recently been reported by Flessas (1981)—hereafter referred to as I. We give below a systematic procedure for determining such solutions which yields not only the solutions of even parity reported in I, but also a similar set of odd parity solutions.

We begin by writing

$$\psi(x)=\exp(-x^2/2)\sum_{n=0}^{\infty} a_n x^{2n+\nu}. \tag{2}$$

Substitution into equation (1) yields an indicial equation with solutions $\nu=0$ or 1 and a three-term recursion relation for the coefficients a_n given by

$$\alpha^{\nu}_{n+2}a_{n+2}+\beta^{\nu}_{n+1}a_{n+1}+\gamma^{\nu}_n a_n=0 \tag{3}$$

with

$$\alpha^{\nu}_n=-(2n+\nu)(2n+\nu-1), \tag{4}$$

$$\beta^{\nu}_n=2(2n+\nu)+1-E+g\alpha^{\nu}_n, \tag{5}$$

$$\gamma^{\nu}_n=[2(2n+\nu)+1-E]g+\lambda. \tag{6}$$

Solutions of the kind obtained in I exist if the infinite series in equation (2) can be made to terminate. For a_m to be the last non-vanishing coefficient in the series for the wavefunction, it is necessary that the next two successive coefficients a_{m+1} and a_{m+2} vanish. Since $a_m\neq 0$ and $a_{m+1}=0$, for $a_{m+2}=0$ it is necessary that $\gamma^{\nu}_m=0$. It is easily seen that the condition $a_{m+1}=0$ is equivalent to the vanishing of the $(m+1)$th

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determinant given by

$$\Delta^{\nu}_{m+1}(g, \lambda, E) = \begin{vmatrix} \beta^{\nu}_0 & \alpha^{\nu}_1 & & & & & 0 \\ \gamma^{\nu}_0 & \beta^{\nu}_1 & \alpha^{\nu}_2 & & & & \\ & \gamma^{\nu}_1 & \beta^{\nu}_2 & \alpha^{\nu}_3 & & & \\ & & & \ddots & & & \\ & 0 & & & \gamma^{\nu}_{m-2} & \beta^{\nu}_{m-1} & \alpha^{\nu}_m \\ & & & & & \gamma^{\nu}_{m-1} & \beta^{\nu}_m \end{vmatrix} . \tag{7}$$

Thus the conditions necessary for the existence of exact solutions of the kind reported in I are

$$\gamma^{\nu}_m = 0, \tag{8}$$

$$\Delta^{\nu}_{m+1} = 0, \tag{9}$$

where $m = 0, 1, 2, \ldots$.

Even parity solutions

It follows from equation (8) that the energy eigenvalues for the even parity solutions ($\nu = 0$) are of the form

$$E = (4m+1) + \lambda/g, \qquad m = 0, 1, 2, \ldots, \tag{10}$$

provided that the condition imposed by equation (9) is also satisfied. We now examine the precise relations that E and g must satisfy for different values of m, in order that exact solutions exist.

(i) For $m = 0$

$$E = 1 + \lambda/g. \tag{11}$$

In addition we must have

$$\Delta^0_1 = \beta^0_0 = 1 - E = 0. \tag{12}$$

Together these imply $\lambda = 0$ and we obtain just the ground state energy of the harmonic oscillator.

(ii) For $m = 1$,

$$E = 5 + \lambda/g, \tag{13}$$

$$\Delta^0_2 = \beta^0_0\beta^0_1 - \alpha^0_1\gamma^0_0 = 0. \tag{14}$$

Together these give

$$(5 - E)(1 - E - 2g) = 0, \tag{15}$$

whose solution is either $E = 5$ and $\lambda = 0$ which yields just the second excited state of the harmonic oscillator, or

$$E = 1 - 2g, \tag{16}$$

$$\lambda = -2(2 + g). \tag{17}$$

Equations (13), (16) and (17) together constitute the first exact solution reported in I.

(iii) For $m = 2$, the two conditions

$$E = 9 + \lambda/g, \tag{18}$$

$$\Delta_3^0 = \beta_0^0(\beta_1^0\beta_2^0 - \alpha_2^0\gamma_1^0) - \beta_2^0\alpha_1^0\gamma_0^0 = 0, \tag{19}$$

together lead to either the fourth excited state of the harmonic oscillator for $\lambda = 0$, or the second exact solution reported in I for

$$\lambda = -7g^2 - 6g \pm g(25g^2 - 12g + 4)^{1/2}. \tag{20}$$

Exact even parity solutions for higher m can be obtained in a similar manner by using equation (10) in conjunction with $\Delta_{m+1}^0 = 0$ and solving the resulting mth order polynomial in E and g (one solution always being $E = 4m + 1$ corresponding to $\lambda = 0$).

Odd parity solutions

The energy eigenvalues of the odd parity states ($\nu = 1$) are of the form

$$E = (4m + 3) + \lambda/g, \qquad m = 0, 1, 2, \ldots. \tag{21}$$

The procedure described above leads, in addition to the harmonic oscillator solutions corresponding to $\lambda = 0$, to the following exact solutions.

(i) For $m = 1$,

$$E = 7 + \lambda/g, \tag{22}$$

$$\lambda = -2g(2 + 3g), \tag{23}$$

with

$$\psi(x) = ax(1 + gx^2)\exp(-x^2/2) \tag{24}$$

where a is an overall normalisation constant.

(ii) For $m = 2$,

$$E = 11 + \lambda/g, \tag{25}$$

$$\lambda = -13g^2 - 6g \pm g(49g^2 - 4g + 4)^{1/2}, \tag{26}$$

$$\psi(x) = a_0x(1 + a_1x^2 + a_2x^4)\exp(-x^2/2), \tag{27}$$

where a_0 is an overall normalisation constant, and

$$a_1 = -(8g + \lambda)/(6g), \tag{28}$$

$$a_2 = 2g(8g + \lambda)/[3(20g^2 + \lambda)]. \tag{29}$$

This procedure can be continued to obtain an infinite set of odd parity wavefunctions and their corresponding eigenvalues provided λ and g satisfy appropriate algebraic conditions.

The results of the present paper taken in conjunction with those reported in I show that for the interaction under consideration there exist an infinite set of exact solutions with eigenvalues of the form

$$E = (2m + 1) + \lambda/g, \qquad m = 0, 1, 2, \ldots, \tag{30}$$

and corresponding wavefunctions given by products of $\exp(-x^2/2)$ and polynomials $P_m(x)$, the polynomials having parity $(-1)^m$.

We conclude by remarking that the procedure outlined in this paper can be easily extended to search for exact solutions of such quantum bound state problems as have k adjustable coupling constants in their Hamiltonians, and which on the substitution of a suitable ansatz for the wavefunction in their corresponding Schrödinger equations lead to recursion relations that involve coefficients whose indices are not separated by more than $(k+1)$ units.

Reference

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